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TRANSIENT TEMPERATURES IN A SEMI-INFINITE CYLINDER HEATED BY A DISK HEAT SOURCE

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Abstract—This paper is part of a series of papers by the author providing the exact transient temperature distribution in bodies heated by disk heat sources. The body in this paper is a semi-infinite cylinder with a uniform disk source centered at the end and uniformly heated. Results are given by infinite series, tables, figures and approximate relations. Care has been taken to provide methods for efficient evaluation of the infinite series because direct evaluation can require thousands of terms. Alternative exact methods are provided that require as few as three terms.

The solution is intrinsically important but it is also a basic building block for spatially and time varying heat fluxes for finite as well as semi-infinite cylinders. This is discussed briefly herein and references for more extensive treatment are provided. The solution is also a basic one for the new numerical procedure called the surface element method.

NOMENCLATURE

a, radius of heated area;
b, radius of cylinder;
c_p, specific heat;
D_l, function defined by equation (23);
$E(\cdot)$, complete elliptic integral of the second kind;
$I(r,z,b)$, function defined by equation (19);
$G_i(r,b)$, function defined by equation (24);
k, thermal conductivity;
$K(\cdot)$, complete elliptic integral of the first kind;
$J_i(\cdot)$, Bessel function of the first kind;
q, heat flux;
r, radial coordinate;
$\mathcal{R}$, a number greater than 2.4; see equation (35);
t, time;
T, temperature;
T_i, initial temperature;
z, axial coordinate.

Greek symbols

α, thermal diffusivity;
β_k, $(2k+1)\pi/2$;
$\Gamma(n)$, gamma function;
$\Gamma(n,x)$, incomplete gamma function;
ε, b^{-1};
ρ, density.

INTRODUCTION

THE TRANSIENT temperature distribution in a semi-infinite cylinder heated over a disk-shaped region centered at the end is a basic problem in heat conduction. The region at $z=0$ for $0 \le r \le a$ is considered to have a constant heat flux and the other surfaces are insulated as shown in Fig. 1.

Analogous problems occur in electric heating, flow in porous media and mass transfer.

The solution can be used as a building block in various problems associated with the contact conductance and temperature corrections for thermocouples embedded in solids and at the surface of solids. For example, a new numerical technique called the 'surface element' method can utilize the solution as a building block [1].

Kennedy, in 1960, presented some analytical solutions and graphical results for the steady state case in finite cylinders [2]. He was interested in thermal and electrical spreading resistance within a package of a semiconductor device. The expression that he gave was evaluated at $r=0$ and $z=0$; for certain other locations his series expression can be very slow to converge with thousands of terms required.

In 1975 Yovanovich [3] presented the steady state portion of the solution for the semi-infinite cylinder. There is no steady state solution to the problem but the general transient solution does have a part that is time-independent and can be related to the contact conductance [3, 4]. The solution provided by Yovanovich for the constant surface heat flux is similar to the one given by Kennedy [2] and the one given herein and can take thousands of terms in its evaluation. However, methods for a more efficient solution are provided herein.

The solution derived herein is of interest for laser heating and drilling [5]. In [5] a transient solution in the form of an integral is given for plates so thin that only radial heat flow is considered.

A similar problem to the one described in this paper was solved by Keltner and Wildin [6] in connection with analysis of foil heat flux gages. An infinite series solution was developed for a finite cylinder insulated at

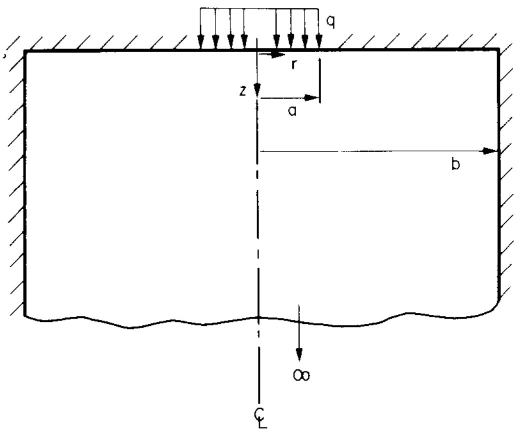

FIG. 1. Semi-infinite cylinder heated over a disk-shaped region centered at $r = 0$ and $z = 0$. Insulated over all other surfaces.

$z = L$ and isothermal at the outer radius b.

Steen [7] utilized finite elements to solve the transient problem including convective and radiant surface heat losses. The motivation was the use of laser heating for surface hardening and surface alloying by vapor deposition.

In the thorough paper of Jury *et al.* [8] the motivation was the investigation of steady state end effects of heat, mass or electricity through a cylindrical rod. Some series solutions with numerical evaluations along with finite different calculations were presented. The steady solution for a finite cylinder that is isothermal at $z = L$ was treated. Their series solution required only tens of terms for evaluation.

The other extreme from steady solutions is the early time behavior. For small dimensionless times the local region in the vicinity of the disk source changes in temperature but the temperatures are negligibly changed beyond the local region. Hence for such early times the solution for a semi-infinite body can be used. The exact solution in terms of an integral is given in Carslaw and Jaeger [9]. Some series solutions that are easier to evaluate have been developed by Beck [10, 11]. Another important integral form of the solution is given in [12].

An important reason that the semi-infinite cylinder is provided herein rather than a finite cylinder is that the former serves as a building block for finite geometries. It is a more basic building block. By superimposing solutions for sources (or sinks) at $z = 0$, $\pm 2L$, $\pm 4L$, ... one can obtain results for finite cylinders that are insulated or isothermal at $z = L$ [10].

MATHEMATICAL DESCRIPTION

The geometry and coordinates are shown in Fig. 1. A mathematical statement of the problem is the solution of

$$k\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{\partial^2 T}{\partial z^2}\right]=\rho c_p\frac{\partial T}{\partial t} \quad (1)$$

$$-k\frac{\partial T(r,0,t)}{\partial z}=\begin{cases} q & \text{for } 0<r<a \\ 0 & \text{for } a<r<b \end{cases} \quad (2a)$$

$$\frac{\partial T(b,z,t)}{\partial r}=0 \quad (2b)$$

$$T(r,z,t)\to T_i \quad \text{for } z\to\infty \quad (2c)$$

$$T(r,z,0)=T_i \quad (2d)$$

where T_i is the initial temperature. For the other symbols, see the nomenclature.

SOLUTION

The solution is developed from the Green's function solution for an instantaneous point source in an infinite insulated cylinder that is given on p. 378 of [9] which can be written as

$$v(r,\theta,z,t')=\frac{e^{-z^2/4\alpha t'}}{2\pi b^2\sqrt{(\pi\alpha t')}}\cdot\left\{1+\sum_{n=-\infty}^{\infty}\cos n(\theta-\theta')\sum_{i=1}^{\infty}e^{-\alpha t'\lambda_i^2}\left[\frac{\lambda_i^2 J_n(\lambda_i r)J_n(\lambda_i r')}{\left(\lambda_i^2-\frac{n^2}{b^2}\right)J_n^2(\lambda_i b)}\right]\right\} \quad (3a)$$

where $(r', \theta', 0)$ is the location of the point source, t' is the time measured from the time of the energy release by the source, and α is the thermal diffusivity. This equation represents the temperature rise for the instantaneous point source. The eigenvalues λ_i are found from

$$J_n'(\lambda_i b)=0, \quad n=0,1,2,\ldots \quad (3b)$$

where the prime on J denotes differentiation with respect to its argument.

For a constant disk source at $z = 0$, the temperature at time t is equal to T_i plus (3a) multiplied by $2qr'\,dr'\,d\theta'\,dt'/\rho c_p$ and integrated from $t' = 0$ to t, $r' = 0$ to a and $\theta' = 0$ to 2π. (The factor 2 is needed because a semi-infinite rather than infinite cylinder is being considered.)

An integral including the θ' dependence is

$$\int_0^{2\pi}\cos n(\theta-\theta')\,d\theta'=\begin{cases} 2\pi & \text{for } n=0 \\ 0 & \text{for } n=1,2,\ldots \end{cases} \quad (4)$$

Consequently only the $n = 0$ term is needed in (3a) and eigenvalues are now found from

$$J_1(\lambda_i b)=0, \quad i=1,2,\ldots. \quad (5)$$

Using the relation

$$\int_0^a r' J_0(\lambda_i r')\,dr'=\frac{a}{\lambda_i}J_1(\lambda_i a) \quad (6)$$

one can obtain

$$T(r,z,t) = T_i + \frac{2qa}{\rho c_p b^2 \sqrt{\pi}} \int_0^t \frac{e^{z^2/(4\alpha t')}}{(\alpha t')^{1/2}} \times \left[\frac{a}{2} + \sum_{i=1}^{\infty} e^{-\alpha t' \lambda_i^2} \frac{J_0(\lambda_i r) J_1(\lambda_i a)}{\lambda_i J_0^2(\lambda_i b)}\right] dt'. \quad (7)$$

The remaining integration is over time. Note that

$$\int_0^t (\alpha t')^{-1/2} e^{-z^2/4\alpha t'} dt' = \frac{2(\pi)^{1/2}}{\alpha} B(z,t) \quad (8)$$

$$\int_0^t (\alpha t')^{-1/2} \exp\left[-\alpha t' \lambda_i^2 - \frac{z^2}{4\alpha t'}\right] dt' = \frac{1}{2} \frac{(\pi)^{1/2}}{\alpha \lambda_i} C_i(z,t) \quad (9)$$

where

$$B(z,t) = (\alpha t)^{1/2} \operatorname{ierfc}[z/2\sqrt{(\alpha t)}] \quad (10)$$

$$C_i(z,t) = e^{-z\lambda_i}\left\{1 + \operatorname{erf}\left[\lambda_i(\alpha t)^{1/2} - \frac{z}{2(\alpha t)^{1/2}}\right]\right\} - e^{z\lambda_i} \operatorname{erfc}\left[\lambda_i(\alpha t)^{1/2} + \frac{z}{2(\alpha t)^{1/2}}\right] \quad (11)$$

where $i = 1, 2, \ldots$. Then (7) can be written as

$$T(r,z,t) = T_i + \frac{2qa}{k}\left\{\frac{aB(z,t)}{b^2} + \sum_{i=1}^{\infty} \frac{C_i(z,t) J_0(\lambda_i r) J_1(\lambda_i a)}{2[\lambda_i b J_0(\lambda_i b)]^2}\right\} \quad (12)$$

which is one form of the result that is being sought.

In order to show the result given by (12) more conveniently it is put into a dimensionless form. Furthermore it is written in a manner to display a time-independent component. Let

$$T^+(r^+, z^+, t^+) \equiv \frac{T(r,z,t) - T_i}{qa/k}, \quad r^+ \equiv \frac{r}{a}, \quad z^+ \equiv \frac{z}{a} \quad (13)$$

$$t^+ \equiv \frac{\alpha t}{a^2}, \quad b^+ \equiv \frac{b}{a}, \quad \lambda_i^+ \equiv \lambda_i b. \quad (14)$$

Using these definitions but dropping the plus superscripts gives

$$T(r,z,t) = \frac{2B(z,t)}{b^2} - \sum_{i=1}^{\infty} \frac{A_i(z,t) J_0(\lambda_i r/b) J_1(\lambda_i/b)}{[\lambda_i J_0(\lambda_i)]^2} + I(r,z,b) \quad (15)$$

where the eigenvalues are found from

$$J_1(\lambda_i) = 0 \quad (16)$$

and now B, A_i and I are given by

$$B(z,t) \equiv t^{1/2} \operatorname{ierfc}(0.5zt^{-1/2}) \quad (17)$$

$$A_i(z,t) \equiv e^{-z\lambda_i/b} \operatorname{erfc}\left(\frac{\lambda_i t^{1/2}}{b} - \frac{z}{2t^{1/2}}\right) + e^{z\lambda_i/b} \operatorname{erfc}\left(\frac{\lambda_i t^{1/2}}{b} + \frac{z}{2t^{1/2}}\right) \quad (18)$$

$$I(r,z,b) \equiv 2 \sum_{i=1}^{\infty} \frac{e^{-z\lambda_i/b} J_0(\lambda_i r/b) J_1(\lambda_i/b)}{[\lambda_i J_0(\lambda_i)]^2}. \quad (19)$$

There are several advantages of writing (15) in the form that is given rather than as the dimensionless form of (12). First it displays clearly the presence of a time-independent term $I(r, z, b)$. For some cases $I(r, z, b)$ gives rise to a steady term. Note that for large times ($t/b^2 > 1$ and $t/z^2 \gg 1$) the explicit sum in (15) goes to zero; for such times

$$T(r,t,z) \approx \frac{1}{b^2}\left[2\left(\frac{t}{\pi}\right)^{1/2} - z\right] + I(r,z,b). \quad (20)$$

This solution can be used to derive a constriction resistance [3]. Second, for a given position (r, z) and b ratio, the function $I(r, z, b)$ need be evaluated only once. This is important because the direct evaluation of $I(r, z, b)$ can involve thousands of terms particularly as b becomes 10 or larger. Third, by focusing attention on this troublesome term some more effective methods of evaluation can be found. It is my conviction that an exact solution is not satisfactory unless it can be evaluated accurately with a moderate number of terms (less than 100, say). Computability is important. Several methods suggested for efficiently evaluating $I(r, z, b)$ are given below.

DISCUSSION OF SOLUTION

Before displaying specific results, several observations are made. For the limiting case of $b = 1$, i.e. uniform heating over the end of the cylinder, the summations given in (15) and (19), are both zero through the use of (16). In this case the temperature distribution is the well-known one-dimensional result [9] of

$$T(z,t) = 2t^{1/2} \operatorname{ierfc}(0.5zt^{-1/2}) \quad (21)$$

which does not exhibit a steady state solution. For the other extreme value of $b \to \infty$ the geometry is a semi-infinite body [10, 11] for which a steady state exists. Provided b is larger than one, the semi-infinite body solution can be used if the value of t/b^2 is sufficiently small.

The explicit summation in (15) converges much more rapidly for $t > 0$ than $I(r, z, b)$ at $z = 0$ (and possibly for all z's). For the small times, $t < 0.02z^2$, the explicit summation is negligibly small. Both summations tend to converge rapidly for $z > 0$, particularly if $z > 0.3b$. Fewer terms are needed in (15) as t becomes large. In fact, for $z < b$ only one term is needed in the summation of (15) if $t > b^2$. For small t's many terms may be needed but then the semi-infinite body solutions [10, 11] can be used.

A location of particular interest is at the surface, $z = 0$, where the dimensionless temperature can be written as

$$T(r,0,t) = \frac{2}{b^2}\left(\frac{t}{\pi}\right)^{1/2} + I(r,0,b) - 2\sum_{i=1}^{\infty} \frac{\operatorname{erfc}(\lambda_i t^{1/2}/b) J_0(\lambda_i r/b) J_1(\lambda_i/b)}{[\lambda_i J_0(\lambda_i)]^2}. \quad (22)$$

Table 1 can be used to obtain numerical comparisons. The denominator in the summation of (22) and in $I(r, z, b)$ is denoted D_i

$$D_i \equiv [\lambda_i J_0(\lambda_i)]^2 \quad (23)$$

which is the second row of Table 1. It monotonically increases with i. The term

$$G_i(r, b) = J_0(\lambda_i r/b) J_1(\lambda_i/b)/D_i \quad (24)$$

is given for $r = 0, 1$ and b for $b = 2, 5$ and 10. For $z = 0$, $G_i(r, b)$ is the ith term in the summation of $I(r, 0, b)$. For $z > 0$, the magnitude of each term in $I(r, z, b)$ is less than the corresponding one in Table 1 for the same r and b values due to the presence of the $\exp(-z\lambda_i/b)$ term in (19). The same is true in the explicit summation of (22) because $\operatorname{erfc}(x)$ is always less than unity for $x > 0$.

As indicated in Table 1 the $|G_i(r, b)|$ terms are always less than unity. As a consequence it is possible for z to be sufficiently large in (19) so that $I(r, z, b)$ is negligible in value; this occurs when

$$e^{-z\lambda_1/b} < 10^{-8} \quad \text{or} \quad z > 9.2b/3.8 \approx 2.4b$$

where 10^{-8} is taken to be close enough to zero. This implies that for $z > 2.4b$ the temperatures are the same as for a uniformly heated end with the same total heat flow as in the disk source. (Only three terms are needed if z is as small as $0.75b$.)

If $z = 0$, the number of terms required to evaluate $I(r, 0, b)$ can run into the thousands. This is particularly true for large b values as suggested by the $G_i(0, 10)$ terms in Table 1. Fortunately there are much more efficient methods to evaluate $I(r, 0, b)$ than using the expression given by (19). Some of these methods are given in connection with (27) to (35).

The explicit summation part of (22) is much easier to evaluate than $I(r, 0, b)$ due to the presence of the $\operatorname{erfc}(\cdot)$ term. For sufficiently large values of $t^{1/2}/b$, the summation is negligibly small and *no* terms are needed.

Table 1. Numerical values of some terms in $I(r, z, b)$. D_i and $G_i(r, b)$ are defined by equations (23) and (24)

	$i = 1$	$i = 2$	$i = 3$
λ_i	3.83171	7.01559	10.17347
D_i	2.38164	4.43308	6.45345
$G_i(0, 2)$	0.24383	0.03025	−0.05207
$G_i(0, 5)$	0.14936	0.12238	0.08898
$G_i(0, 10)$	0.07897	0.07436	0.06905
$G_i(1, 2)$	0.06647	−0.01153	0.00775
$G_i(1, 5)$	0.12822	0.06916	0.01815
$G_i(1, 10)$	0.07610	0.06549	0.05231
$G_i(2, 2)$	−0.09820	0.00908	0.01300
$G_i(5, 5)$	−0.06016	0.03673	−0.02222
$G_i(10, 10)$	−0.03181	0.02232	−0.01724

This is because $\operatorname{erfc}(x)$ decreases very rapidly with x for $x > 2$; for example, $\operatorname{erfc}(x)$ is less than 10^{-7} for x being greater than 3.8. The first λ_i value is about 3.83. Consequently no terms are needed (to within an error of 10^{-7}) for $t^{1/2}/b > 1$ or

$$t > b^2 \quad (25a)$$

which is a convenient expression. If only three terms are used, then necessarily

$$t > b^2/4. \quad (25b)$$

For only somewhat smaller ts than this, the semi-infinite solution is valid [11].

At this point the difficulty in utilizing the solution given by (22) is in evaluating $I(r, 0, b)$. After presenting some results $I(r, 0, b)$ will be considered further.

GRAPHICAL AND TABULAR RESULTS

The temperature distribution at the surface is illustrated in Fig. 2 where the dimensionless temperature is plotted versus dimensionless radial position for the dimensionless times of 0.01, 0.1, 1, 10 and 100. Some of the same information is contained in Table 2 and 3. Notice that the $t = 0.01$ curve is very flat over the heated area until $r \approx 0.9$ and then drops to about zero beyond $r \approx 1.1$. For such early times the temperature distribution over the heated area is almost identical to that obtained in a semi-infinite body heated uniformly over its surface. The temperature distribution from (21) reduces to

$$T(0, t) = 2(t/\pi)^{1/2}. \quad (26)$$

Also for such a small dimensionless time nearly the same surface temperature distribution would be found for any b value greater than about 1.1. Somewhat similar behavior is noted for $t = 0.1$ but the uniform temperature region is now smaller. In addition, the temperature rise extends out to $r \approx 1.8$; consequently until $t = 0.1$ the temperatures for $r \geq 0$ are identical for all bs greater than 1.8. In other words, the temperatures for $t \geq 0.1$ and $0 < r < b < 1.8$ are identical to those obtained in a semi-infinite body heated by a disk source [11]. For a more precise comparison the early time values given in Tables 2 and 3 can be compared with those tabulated in [11]. The $b = 10$ values provided in Table 3 deviate only in the sixth place for $t = 0.1$ from the semi-infinite values [11].

The effect of the boundary at $r = b$ is felt at 'large' times. In Fig. 2 two $t = 10$ and $t = 100$ curves are shown with the solid curve representing $b = 10$ and the dashed curve $b = 5$. The temperatures are increased due to the presence of the insulated boundary at $r = b$.

Another way to look at the surface temperature is to examine plots of specific locations as a function of r. See Fig. 3 which is a log–log plot of T vs t for $r = 0, 1, 2$ and 10 for $b = 10$. The straight line on the upper left represents the surface temperature of a uniformly heated semi-infinite body ($b = 1$). At the center ($r = 0$) and for $t < 0.1$ the T curve is nearly the same as if the

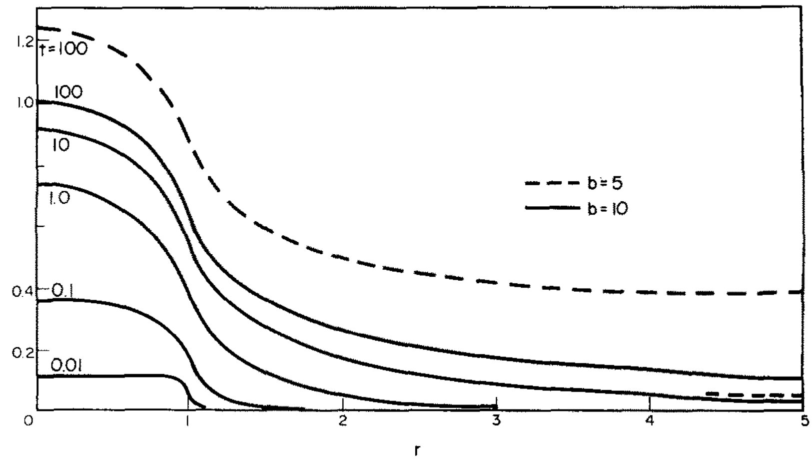

FIG. 2. Dimensionless local temperature distribution at $z = 0$ for various dimensionless times as a function of dimensionless radius.

Table 2. Local transient temperatures for $b = 2$ for various radii

Time, t	Temperatures for various radii values $r = 0.0$	$r = 0.5$	$r = 1.0$	$r = 1.5$	$r = 2.0$
0.01	0.112838	0.112836	0.054825	0.000001	0.0
0.1	0.352882	0.336874	0.162280	0.009498	0.000766
0.5	0.631692	0.578342	0.314891	0.083427	0.047691
1.0	0.738065	0.679560	0.403971	0.160964	0.120769
5.0	1.090040	1.030823	0.753556	0.508948	0.468133
10.0	1.351319	1.292102	1.014835	0.770226	0.729412
50.0	2.453968	2.394751	2.117484	1.872876	1.832062
100.0	3.280205	3.220988	2.943720	2.699112	2.658298

Table 3. Local transient temperatures for $b = 10$ for various radii

Time, t	Temperatures for various radii values $r = 0$	$r = 0.5$	$r = 1.0$	$r = 1.5$	$r = 2.0$
0.1	0.352882	0.33687	0.16228	0.009490	0.000371
1.0	0.729097	0.668299	0.38471	0.124565	0.051148
10.0	0.911166	0.845566	0.54851	0.268722	0.172673
50.0	0.969185	0.903452	0.60600	0.325571	0.228648
100.0	1.002254	0.936521	0.63907	0.358639	0.261715
500.0	1.141730	1.075997	0.77855	0.498114	0.401190
800.0	1.208570	1.142837	0.84539	0.564954	0.468031
1000.0	1.246241	1.180508	0.88306	0.602626	0.505702

body were uniformly heated. At $r = 1$ and for $t < 0.01$ the temperature rise is almost exactly one-half that given at $r = 0$. The above comments apply for any $b \geq 2$. For the large times ($t > 10^4$) the temperatures again vary linearly in the log–log plot and demonstrate that no steady state condition exists for finite b.

Figure 4 is similar to Fig. 3 but the curves are for the center location of $r = 0$, $z = 0$. Several b values are shown. The $b = 1$ curve is for a completely heated surface; the geometry can be either a semi-infinite body or a semi-infinite cylinder. The $b = 2$, 5 and 10 curves are for a semi-infinite cylinder and $b \to \infty$ corresponds to a semi-infinite body heated only over a disk-shaped region. A steady state condition exists only for the latter case. For the finite b's a quasi-state condition exists.

Because the temperatures rise indefinitely with time for finite b values but with a quasi-steady state being attained, it is instructive to plot

$$T(r,0,t) - \frac{2}{b^2}\left(\frac{t}{\pi}\right)^{1/2}$$

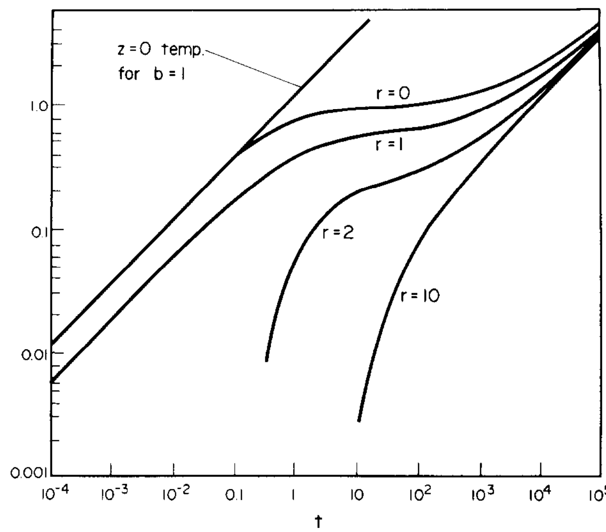

FIG. 3. Dimensionless centerline temperature at $z = 0$ for several b values.

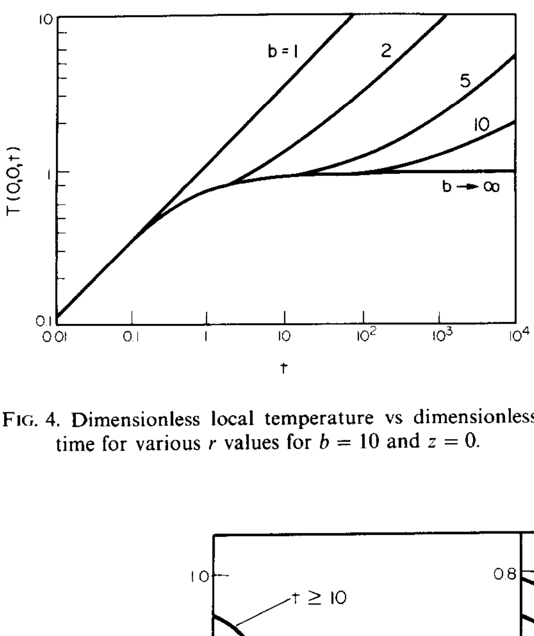

FIG. 4. Dimensionless local temperature vs dimensionless time for various r values for $b = 10$ and $z = 0$.

versus r for various times. Results for $b = 2$, 5 and 10 are depicted in Fig. 5. The approach to a quasi-steady condition is shown to occur in the least time for the smallest b value. The quasi-steady state values in Fig. 5 are simply $I(r, 0, b)$ which illustrates the importance of the latter term.

VALUES OF $I(r, 0, b)$

The function $I(r, 0, b)$ is shown in two different ways. Table 4 gives values as a function of r/b and $b^{-1} = \varepsilon$. Figure 6 shows $I(r, 0, b)$ versus ε for fixed r values. In this figure the $I(r, 0, b)$ curves are seen to be almost linear with ε for small values of ε.

The $\varepsilon = 0$ (or equivalently $b \to \infty$) case corresponds to the physical geometry of a semi-infinite body [11]. For this case of $b \to \infty$

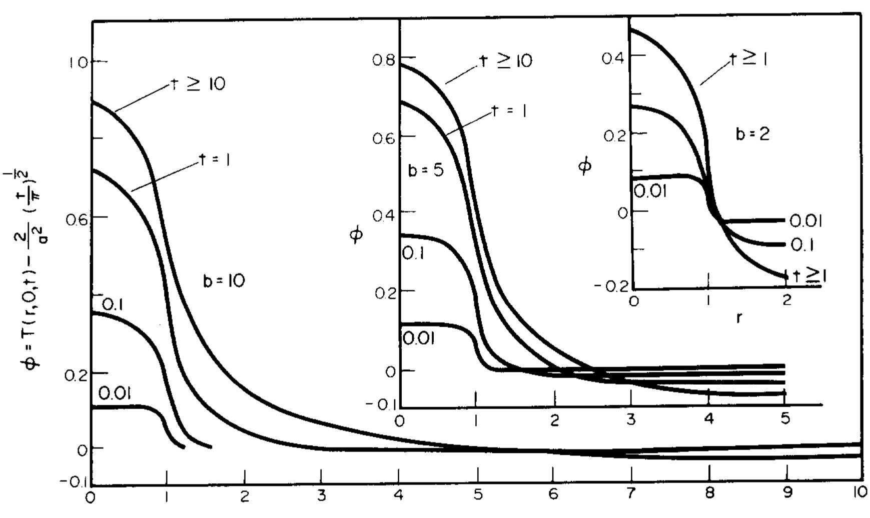

FIG. 5. Dimensionless local temperature minus $(2a^{-2})(t/\pi)^{1/2}$ vs dimensionless radius for various times and $b = 2$, 5 and 10.

Table 4. The summation $I(r, 0, b)$ as a function of r/b and ε ($\varepsilon = 1/b$)

r/b	$\varepsilon = b^{-1}$ values: 0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0	1.0	0.88942	0.77943	0.67066	0.56371	0.45924	0.35793	0.26051	0.16778	0.08061	0
0.1	0	0.52624	0.71405	0.64288	0.54872	0.45021	0.35224	0.25693	0.16568	0.07966	0
0.2	0	0.14888	0.41767	0.55043	0.50123	0.42216	0.33471	0.24598	0.15930	0.07676	0
0.3	0	0.06031	0.13901	0.31280	0.41054	0.37153	0.30383	0.22692	0.14827	0.07177	0
0.4	0	0.01866	0.04462	0.09003	0.21371	0.28883	0.25621	0.19825	0.13188	0.06440	0
0.5	0	−0.00496	−0.00601	0.00202	0.02889	0.12277	0.18362	0.15695	0.10883	0.05415	0
0.6	0	−0.01951	−0.03649	−0.04796	−0.04958	−0.03295	0.04294	0.09593	0.07661	0.04012	0
0.7	0	−0.02869	−0.05551	−0.07837	−0.09464	−0.10041	−0.08813	−0.02183	0.02919	0.02050	0
0.8	0	−0.03431	−0.06708	−0.09663	−0.12102	−0.13773	−0.14302	−0.12974	−0.06545	−0.00926	0
0.9	0	−0.03732	−0.07326	−0.10632	−0.13485	−0.15685	−0.16966	−0.16933	−0.14843	−0.07607	0
1.0	0	−0.03826	−0.07516	−0.10930	−0.13909	−0.16265	−0.17760	−0.18061	−0.16627	−0.12359	0

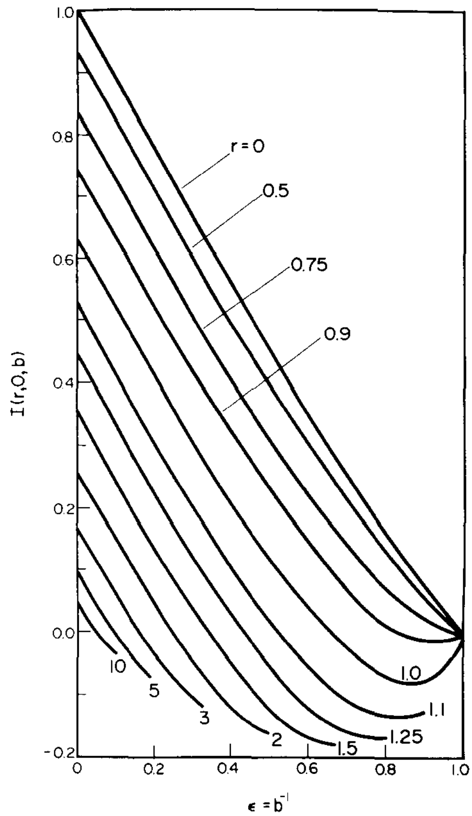

FIG. 6. $I(r, 0, b)$ vs $\varepsilon = b^{-1}$ for fixed radii values.

$$I(r, 0, \infty) = \frac{2}{\pi} E(r), \quad 0 \le r \le 1 \tag{27a}$$

$$= \frac{2r}{\pi}\left[E\left(\frac{1}{r}\right) - (1 - r^{-2}) K\left(\frac{1}{r}\right)\right], \quad r \ge 1 \tag{27b}$$

where $K(\cdot)$ and $E(\cdot)$ are the complete elliptic integrals of the first and second kinds, respectively. For large r values (27b) can be approximated by

$$I(r, 0, \infty) \approx \frac{1}{2r}. \tag{28}$$

The linearity of I with ε in Fig. 6 suggests the approximation of

$$I(r, 0, b) \approx I(r, 0, \infty) - 1.106824/b \tag{29}$$

which is valid for $b > 10$. This is interesting in that it indicates that the difference $I(r, 0, b) - I(r, 0, \infty)$ does not depend on r. The approximation provided by (29) is excellent provided $r < b/4$.

For $r = 0$ and any ε ($= b^{-1}$), $I(0, 0, \varepsilon)$ can be approximated by

$$I(0, 0, \varepsilon) = 1 - 1.106824\varepsilon - 0.0007172\varepsilon^2 + 0.1038745\varepsilon^3 - 0.0084825\varepsilon^4 + 0.0121492\varepsilon^5 \tag{30}$$

which is accurate to six decimal places. At the extreme r value, namely b, and for large b's one can use

$$I(b,0,b) \approx -0.38479b^{-1} + 0.02225b^{-2} \quad (31)$$

which is accurate to four significant figures for $b > 5$.

EVALUATION OF $I(r, 0, b)$ USING SERIES

There are several ways to reduce the computation load in evaluating $I(r, 0, b)$ from series expressions. Two of these utilize known transient solutions and a third employs another equivalent series.

The first of these methods evaluates $I(r, 0, b)$ by equating (15) to the known solutions of

$$T(r,0,t) = 2(t/\pi)^{1/2}, \quad 0 \le r < 1 \quad (32a)$$

$$T(1,0,t) = (t/\pi)^{1/2} \quad (32b)$$

$$T(r,0,t) \approx 0, \quad r > 1 \quad (32c)$$

provided t is sufficiently small such as 0.0001. The resulting equation is solved for $I(r, 0, b)$. Rather than thousands of terms only hundreds are needed.

The same idea can be employed with the semi-infinite body solution [11] with times large compared to 0.0001 but still small enough so that the temperatures at the point of interest are changing as though the body were semi-infinite. Using this approach with $T(r, 0, t)$ in (22) replaced by the semi-infinite solution denoted $T_{s.i.}(r, 0, t)$ yields

$$I(r,0,b) \approx 2\sum_{i=1}^{i_{max}} \frac{\operatorname{erfc}(\lambda_i t^{1/2}/b) J_0(\lambda_i r/b) J_1(\lambda_i/b)}{[\lambda_i J_0(\lambda_i)]^2} + T_{s.i.}(r,0,t) - \frac{2}{b^2}\left(\frac{t}{\pi}\right)^{1/2}. \quad (33)$$

Any time t satisfying

$$t \le 0.01b^2 \quad (34)$$

can be chosen to solve (33); this criterion applies to any r. If r is small, say $r \le 1$, then even larger times are possible such as

$$t \le b^2 A$$

where $A = 0.025$ for $b = 2$, $A = 0.04$ for $b = 5$ and $A = 0.1$ for $b = 10$. The largest permissible time in (33) reduces the required number of terms, i_{max}, in (33) to only 3 or 4. Clearly this means a tremendous reduction in computation though $T_{s.i.}(r, 0, t)$ must still be evaluated. Fortunately the latter can also be calculated with a small number of terms [11].

The final method to be mentioned utilizes another series expression that is given in [8]. In this paper the steady state solution is given for a finite cylinder maintained at $z = L$. As pointed out below equation (24), the effect of the disk source has been smoothed out by the z/b ratio of about 2.4. The present solution can be superimposed (utilizing sources and sinks) to obtain the result for a finite cylinder for steady state and transient conditions. If the length L of the cylinder is taken to be $2.4b$ (or greater), the only source (or sink) having a significant contribution is the source at $z = 0$.

Utilizing the solution given by Jury *et al.* [8] it can be shown that for $r > 1$

$$I(r,0,b) = \frac{\mathcal{R}}{b}\left(\frac{z}{\mathcal{R}b} - 1\right) + \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{(-1)^k \sin\left[\beta_k\left(1 - \frac{z}{\mathcal{R}b}\right)\right] I_1\left(\frac{\beta_k}{\mathcal{R}b}\right)\left[K_1\left(\frac{\beta_k}{\mathcal{R}}\right) I_0\left(\frac{\beta_k r}{\mathcal{R}b}\right) + I_1\left(\frac{\beta_k}{\mathcal{R}}\right) K_0\left(\frac{\beta_k r}{\mathcal{R}b}\right)\right]}{(2k+1) I_1(\beta_k/\mathcal{R})} \quad (35)$$

where $\mathcal{R} = 2.4$ or larger and $\beta_k = (2k + 1)\pi/2$. The larger $\mathcal{R}$ is made, the more terms are needed. For all r values somewhat greater than one, (35) can be efficiently evaluated. For example, for $b = 10$, $r = 5$ and $z = 0$ only 16 terms are needed.

Though the series expression is relatively straightforward to evaluate, mistakes can be made. When Jury *et al.* [8] utilized this series for $r = b$ and $z = 0$ for an electrical heater problem, they made some computational errors. In their Table I they gave a voltage from double precision calculations as 19.6552396 ... while their finite difference calculation for the same point was 19.62678. Carefully using their expression gives the value of 19.616009 which shows that their finite difference value was more accurate than their 'exact' result. This coincides with my previous experience that 'exact' series expressions are not easy to evaluate even when the required number of terms is not large. That is why it is helpful to have some correct values available as (I believe) are given in Table 4.

In conclusion, the simplest way to evaluate $I(r, 0, b)$ is to use Table 4 if the needed values are contained therein. Interpolation is possible throughout most of the table. For large b values (29) is very convenient and (30) is good for all b's for $r = 0$. For $r = b$ and large b, (31) can be employed. For any r and b, the series given by (33) is very efficient although it requires the semi-infinite body solution [11]. Since the semi-infinite body solution for the times required usually needs fewer than ten terms, the method based on (33) is recommended.

SUMMARY

An exact series solution is developed for a semi-infinite cylinder insulated on the sides and with a disk source centered at the end. Graphs and tables are provided to provide insight into the solution. The term in the exact solution, $I(r, z, b)$, can take thousands of terms to evaluate in a direct fashion. Much more efficient methods are provided.

The solution is a basic one in heat conduction. It can be utilized to obtain many related solutions for steady state as well as transient cases, finite as well as semi-infinite cylinders [10], and for various space and time

variations of the surface heat flux [3, 10, 11]. The given solution can be considered to be a building block for many other cases. It can be utilized for various complex connected bodies using the surface element method which is currently under development.

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TEMPERATURES VARIABLES DANS UN CYLINDRE SEMI-INFINI, CHAUFFE PAR UNE SOURCE DE CHALEUR EN DISQUE

Résumé—L'article est une partie d'un ensemble de textes, par l'auteur, donnant la distribution exacte de température variable dans des corps chauffés par des sources de chaleur en disque. Ici, il s'agit d'un cylindre semi-infini avec un disque-source centré à l'extrémité et uniformément chauffé. Les résultats sont donnés sous forme de séries infinies, de tables, de figures et de relations approchées. Une attention est portée aux méthodes pour l'évaluation de l'efficacité des séries infinies car une évaluation directe peut nécessiter un millier de termes. Des méthodes exactes alternées sont fournies qui demandent à peine trois termes.

La solution est intrinsèquement importante mais elle constitue aussi une base pour les flux thermiques variables aussi bien pour les cylindres finis que semi-infinis. Ceci est discuté brièvement et des références pour un traitement sont fournies. La solution est aussi une base pour la procédure numérique nouvelle appelée la méthode des éléments de surface.

INSTATIONÄRE TEMPERATURVERTEILUNG IN EINEM HALBUNENDLICHEN ZYLINDER, DER VON EINER KREISFÖRMIGEN WÄRMEQUELLE BEHEIZT WIRD

Zusammenfassung—Der Aufsatz gehört zu einer Serie von Arbeiten des Autors zur exakten instationären Temperaturverteilung in Körpern, die von kreisförmigen Wärmequellen beheizt werden. Im vorliegenden Fall ist der Körper ein halbunendlicher Zylinder mit einer gleichförmigen Quelle in Form einer Kreisfläche und gleichmäßiger Beheizung. Die Ergebnisse werden als unendliche Reihen, Tabellen, Diagramme und Näherungslösungen angegeben. Es wurde Wert auf effiziente Auswertungsmethoden für die unendlichen Reihen gelegt, weil die direkte Auswertung die Berücksichtigung Tausender von Gliedern erfordern kann. Geänderte Methoden werden angegeben, die z. B. nur drei Glieder erfordern. Die Lösung an sich ist bereits wichtig, sie stellt aber darüberhinaus auch eine Grundlösung für den zeitlich und örtlich veränderlichen Wärmefluß sowohl in endlichen als auch halbunendlichen Zylindern dar. Hierauf wird kurz eingegangen, und Hinweise für eine ausführlichere Behandlung werden gegeben. Die Lösung ist auch Grundlage eines numerischen Verfahrens mit der Bezeichnung "Oberflächenelement-Methode".

НЕСТАЦИОНАРНОЕ ПОЛЕ ТЕМПЕРАТУР В ПОЛУБЕСКОНЕЧНОМ ЦИЛИНДРЕ, НАГРЕВАЕМОМ ТЕПЛОВЫМ ИСТОЧНИКОМ В ФОРМЕ ДИСКА

Аннотация — Предлагаемая статья является частью серии работ автора по исследованию нестационарного распределения температур в телах, нагреваемых тепловыми источниками в форме диска. В данной статье рассматривается полубесконечный цилиндр, на торце которого по центру расположен равномерно распределенный источник тепла в форме диска. Результаты представлены в виде бесконечных рядов, таблиц, рисунков и приближенных соотношений. Особое внимание уделено обоснованию методов расчета бесконечных рядов, так как для прямого расчета требуется огромное число членов ряда. Представлены также другие точные методы, требующие вычисления не более трех членов.

Полученное решение не только представляет интерес само по себе, но и служит основой расчета нестационарного нагрева конечных и полубесконечных цилиндров. Этот вопрос кратко обсуждается в статье и приводятся ссылки на источники, в которых дается более полный анализ. На основе решения разработан также новый численный метод, называемый методом элементарных поверхностей.